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THE INVERSE OF CONICS AND CONICOIDS FROM THE CENTER.

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For the purposes of this paper it is convenient first to discuss briefly the locus of a point which moves so that the sum, or difference, of its distances from two fixed points bears a constant ratio to its distance from a third point midway between these two.

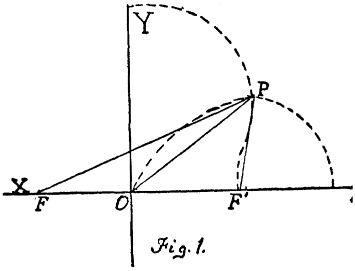

Fig. 1.

In Fig. 1 let P be the moving point, F, F' and O the three fixed points. Take O for origin, and the line FF' for the axis of abscissae; then the condition is

$$\frac{PF \pm PF'}{PO} = \text{a constant.} \tag{1}$$

This gives the equation,

$$[(x + OF)^2 + y^2]^{\frac{1}{2}} \pm [(OF - x)^2 + y^2]^{\frac{1}{2}} = k\,(x^2 + y^2)^{\frac{1}{2}} \tag{2}$$

For convenience let $OF = OF' = \frac{a}{e}$, and $k = \frac{2}{e}$; equation (2) becomes, after clearing of radicals:

$$a^2 x^2 + y^2 \frac{a^2}{1 - e^2} = (x^2 + y^2)^2 \tag{3}$$

It is evident from the form of this equation that the curve is symmetrical about either axis; and that as e varies the shape of the curve will change accordingly. Hence e may conveniently be called the *eccentricity* of the curve; and the fixed points F and F', *foci.*

Denote the numerical value of

$$\frac{a^2}{1 - e^2} \text{ by } b^2, \text{ whence } e^2 = \frac{\pm b^2 - a^2}{\pm b^2} \tag{4}$$

Equation (3) may now be written in the symmetrical form,

$$a^2 x^2 \pm b^2 y^2 = (x^2 + y^2)^2 \tag{5}$$

according as $e < 1$ or $e > 1$.

Consider first the case when $e < 1$, and the equation is,

$$a^2 x^2 + b^2 y^2 = (x^2 + y^2)^2 \tag{6}$$

The curve cuts the axes of coordinates at the points $(\pm a, o)$, $(k, \pm b)$; and as it is a closed curve, a may be called the minor, and b the major semi-axis respectively.

The distance from center to either focus is $\frac{a}{e}$. Since the equation contains no terms of a lower degree than the second, the origin is a conjugate point. Also the equation of imaginary tangents at the origin is

$$a^2 x^2 + b^2 y^2 = 0. \tag{7}$$

The circular points at infinity are imaginary double points for, making the given equation homogeneous by means of the line at infinity, $0x + 0y + c = 0$, gives

$$(x^2 + y^2)^2 = 0, \tag{8}$$

which is the equation of the lines through the origin and the points common to the curve and the line at infinity. But this breaks up into

$$(x + iy)^2 (x - iy)^2 = 0; \tag{9}$$

that is, the four lines reduce to two pairs of coincident lines, showing that the line at infinity cuts the curve in two imaginary double points.

The curve has four foci, two real and two imaginary: for the foci of a curve are determined by the points of intersection of tangents from the circular points at infinity. In the case of a curve of the fourth degree there are in general twelve such tangents; but as the circular points at infinity are also double points, there are in this case but four distinct tangents, which intersect in two real and two imaginary points. The two real points are the foci F, F' mentioned above.

When $e = 0$, $a = b$, and the equation breaks up into the two factors,

$$x^2 + y^2 = a^2, \text{ and } x^2 + y^2 = 0; \tag{10}$$

that is, the curve is a circle of radius a and a point circle at the origin. When $e > 0$ and $< \sqrt{\tfrac{1}{2}}$, the curve is a smooth oval. When $e = \sqrt{\tfrac{1}{2}}$, it changes to an indented oval. In order to determine when this change occurs, it is only necessary to express the condition that the curvature at the x axis shall be zero. Using the polar equation

$$r^2 = a^2 \cos^2 k + b^2 \sin^2 k, \tag{11}$$

the initial line being the axis of abscissæ, the condition for zero curvature at a point is:

$$r^2 + 2\frac{dr^2}{dk^2} - r\frac{d^2r}{dk^2} = 0, \tag{12}$$

which becomes, when values are substituted:

$$\begin{gathered} a^2 \cos^2 k + b^2 \sin^2 k - (b^2 - a^2) \cos 2k \\ + \tfrac{3}{4} (b^2 - a^2)^2 \sin^2 2k\, (a^2 \cos^2 k + b^2 \sin^2 k)^{-1} = 0 \end{gathered} \tag{13}$$

When $k = 0$ this reduces to $2a^2 = b^2$, whence $e = \sqrt{\tfrac{1}{2}}$, the required condition.

When $e > \sqrt{\tfrac{1}{2}}$ and $e < 1$, the curve is an indented oval, with four real points of inflection. In order to determine these points of inflection, set

$$\frac{d^2u}{dk^2} + u = 0, \tag{14}$$

where $u = \dfrac{1}{r}$ of equation (11).

This gives, after substituting values and reducing,

$$k = \tan^{-1} \pm \frac{a}{b} \left[\frac{b^2 - 2a^2}{2b^2 - a^2}\right]^{\frac{1}{2}} \tag{15}$$

Equation (15) shows that there are four points of inflection symmetrically placed about the origin, as might have been inferred from the form of the curve. Substituting the values of $\sin^2 k$ and $\cos^2 k$ obtained from equation (15), in equation (11), the radius vector to the points of inflection is found to be

$$r = \frac{\pm a b \sqrt{3}}{\sqrt{2}\sqrt{a^2 + b^2}} \tag{16}$$

showing that the four points of inflection lie on a circle concentric with the curve.

When $e = 0$; $b = a$ and $\tan k$ is imaginary; showing that there are no real points of inflection.

When $e = \sqrt{\tfrac{1}{2}}$; $b^2 = 2a^2$, $\tan k = 0$, and $r = \pm a$, showing that the four points of inflection coincide two and two at the extremities of the minor axis. This is the transition from a smooth to an indented oval. When $e = 1$; $a = 0$, $\tan k = 0$, $r = 0$; that is, the points of inflection are all at the origin.

In this case the equation of the curve breaks up into

$$x^2 + y^2 = by \text{ and } x^2 + y^2 = -by, \tag{17}$$

two circles of radius $\tfrac{1}{2}b$, tangent to the axis of abscissæ at the origin. They will be finite if e becomes unity by a being zero; infinite, and hence simply two straight lines coincident with the axis of abscissæ, if e is made unity by b assuming a value that is infinite compared with a.

The length, S, of the curve may be represented by the definite integral,

$$\tfrac{1}{4}S=\int_{0}^{\frac{\pi}{2}}\left[r^{2}+\frac{dr^{2}}{dk^{2}}\right]^{\frac{1}{2}}dk, \tag{18}$$

which becomes, after values are substituted, $\tfrac{1}{4}S=\int_{0}^{\frac{\pi}{2}}\left[\frac{a^{4}\cos^{2}k+b^{4}\sin^{2}k}{a^{2}\cos^{2}k+b^{2}\sin^{2}k}\right]^{\frac{1}{2}}dk.$

The area, A, of the curve may be found by the formula

$$\tfrac{1}{4}A=\int_{0}^{\frac{\pi}{2}}r^{2}\,dk, \tag{19}$$

which gives for the whole area inclosed by the curve the value, $\frac{\pi}{2}(a^{2}+b^{2})$.

The inverse of the given curve with respect to the center is,

$$a^{2}x^{2}+b^{2}y^{2}=1, \tag{20}$$

an ellipse whose eccentricity is identical with that of the given curve. The polar reciprocal of this ellipse is,

$$\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1. \tag{21}$$

If circles be described upon the semi-diameters of this latter ellipse, they will envelop the quartic under discussion. Hence, in general, the inverse of an ellipse from the center is the pedal curve of its polar reciprocal with respect to the center.

The above relations furnish a convenient method of investigating many properties of the curve given at the beginning of this paper.

The curve may be described mechanically by means of its pedal property, thus:

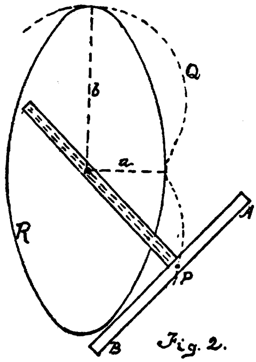

Fig. 2.

Let R be an elliptical board with semi-axis a and b, a "T" square is held by a pivot at O working in a slot, while the arm AB slides against the board; a pencil or crayon at P will describe the required curve Q.

Following is a partial list of theorems relating to the curve. They were obtained by inverting those properties only of the ellipse which in some way were dependent upon the center of the curve. The first paragraph contains the original theorem in the conic; the second paragraph, that relating to the quartic.

From the fact that straight lines invert into circles passing through the origin, it is convenient to call all such circles "central" circles. The quartic will be called an oval.

1. Two tangents can be drawn to an ellipse from any point, which will be real, coincident, or imaginary, according as the point is outside, upon or within the curve.

2. Two "central" tangent circles can be drawn to an oval from any point, which will be real, coincident, or imaginary, according as the point is within, upon, or outside the curve.

1. If the polar of a point P with respect to an ellipse pass through the point Q, then will the polar of Q pass through P.

2. If the "central" polar circle of a point P with respect to an oval pass through the point Q, then will the "central" polar circle of Q pass through P. [By "central" polar circle is meant the circle passing through the origin and the points of contact of "central" tangent circles from the point to the oval.]

1. The locus of the point of intersection of two tangents to an ellipse, which are at right angles to one another, is the director circle of radius equal to $(a^2 + b^2)^{\frac{1}{2}}$ when a and b are the semi-axes respectively.

2. The locus of the point of intersection of two "central" tangent circles to an oval which cut one another orthogonally is a circle of radius $\frac{a\,b}{(a^2 + b^2)^{\frac{1}{2}}}$ when a and b are the semi-axes of the oval respectively.

1. The equation of the locus of the foot of the perpendicular from the center of an ellipse on a tangent is $r^2 = a^2\, cos^2\, k + b^2\, sin^2\, k$; the equation of the ellipse being $1 = \frac{r^2\, cos^2\, k}{a^2} + \frac{r^2\, sin^2\, k}{b^2}$.

2. The equation of the locus of the extremity of the diameter through the origin of a "central" tangent circle to the oval given by the equation $r^2 = a^2\, cos^2\, k + b^2\, sin^2\, k$ is an ellipse whose equation is $1 = \frac{r^2\, cos^2\, k}{a^2} + \frac{r^2\, sin^2\, k}{b^2}$.

1. The sum of the reciprocals of the squares of any two diameters of an ellipse which are at right angles to one another is constant.

2. The sum of the squares of any two diameters of an oval which are at right angles to one another is constant.

1. The line joining the extremities of any two diameters of an ellipse which are at right angles to one another will always touch a fixed circle.

2. The "central" circle joining the extremities of any two diameters of an oval which are at right angles to one another will always touch a fixed circle.

1. The tangent at a point P of an ellipse meets the tangent at A, one extremity of the axis ACA', in the point Y; then is CY parallel to $A'P$, C being the center of the curve.

2. The "central" tangent circle at a point P of an oval cuts the "central" tangent circle at A, one extremity of the axis ACA', in the point Y; then is C Y tangent at C to the "central" circle through P and A', C being the center of the curve.

1. If three of the sides of a quadrilateral inscribed in an ellipse are parallel respectively to three given straight lines, then will the fourth side also be parallel to a fixed straight line.

2. Four points are taken on an oval: if three of the four "central" circles through the consecutive pairs of these points have their "central diameters" perpendicular to three given lines respectively, then will the "central diameter" of the fourth "central" circle be perpendicular to a given line. [By "central diameter" of a "central" circle is meant that diameter which passes through the origin.]

1. A parallelogram circumscribes a circle, and two of the angular points are on fixed straight lines parallel to one another and equidistant from the center; then are the other two on an ellipse of which the circle is the minor auxiliary circle.

2. Four "central" circles are inscribed in a circle so that the two opposite ones have a common "central diameter"; two of the four points of intersection

lie on equal fixed "central" circles having a common "central diameter" and on opposite sides of the center; then are the other two points of intersection on an oval to which the circle is tangent at the extremities of the major axis.

1. Three points, A, P, B, are taken on an ellipse whose center is C. Parallels to the tangents at A and B drawn through P meet CB and CA respectively in the points Q and R. Then is QR parallel to the tangent at P.

2. Three points, A, P, B, are taken on an oval whose center is C. "Central" circles drawn through P and having common "central diameters" with the "central" tangent circles at A and B respectively, meet the lines CB and CA respectively in the points Q and R. Then the "central" circle through Q and R has a common "central diameter" with the "central" tangent circle at P.

1. The sum of the distances from any point on an ellipse to the two foci is constant. The ellipse may be described mechanically by the use of this property.

2. The sum of the distances from any point on an oval to the two foci bears a constant ratio to its distance from the center. The oval, also, may be described mechanically by the use of this property.

1. To draw a tangent at any point of an ellipse, bisect the external angle between the focal radii: to draw a normal, bisect the interior angle.

2. To draw a tangent line to an oval, bisect the external angle formed by the two "central" circles through the point and the two foci respectively: to draw a normal, bisect the interior angle.

1. The subtangent of an ellipse is equal to the corresponding subtangent of the circle described upon the major axis.

2. Given an oval and a circle described upon its minor axis. "Central" tangent circles are drawn at the points where the oval and circle are cut by a "central" circle that intersects the major axis at right angles. Then will the two "central" tangent circles cut the transverse axis at the same point.

Consider now the case where $e > 1$, and the equation of the curve is

$$a^2 x^2 - b^2 y^2 = (x^2 + y^2)^2 \tag{6'}$$

The development of the properties of this curve is much the same as for the quartic first considered. Analytically, it is only necessary to change b^2 to $-b^2$ in the equations of the former to obtain the corresponding equations in the latter.

The curve cuts the axis of coördinates at the points $(\pm a, 0)$ $(0, 0)$; and a and b may be called the minor and major semi-axis respectively.

The distance from center to either focus is $\frac{a}{e}$.

The origin is a real double point. The equation of real tangents at the origin is,

$$a^2x^2 - b^2y^2 = 0. \tag{7'}$$

The circular points at infinity are imaginary double points.

The curve has four foci, two real and two imaginary.

The origin is a point of inflection.

When $e = 1$; $a = 0$; and the equation breaks up into two imaginary circles,

$$x^2 + y^2 = \pm b\, y \sqrt{-1}. \tag{17'}$$

When $e > 1$, the shape of the curve is that of a figure 8 extending along the axis of abscissæ.

When $e = \sqrt{2}$; $a = b$, and the curve is identical with the lemniscate of Bernouilli.

When $e = \infty$; $b = 0$, and the equation breaks up into

$$x^2 + y^2 = \pm ax; \tag{17''}$$

that is, two circles of radius $\frac{1}{2}a$, tangent to the axis of ordinates at the origin.

The length S of the curve may be represented by the definite integral,

$$\tfrac{1}{2}\,S = \int_0^{tan^{-1}\frac{a}{b}} \left(\frac{a^4\,cos^2\,k - b^4\,sin^2\,k}{a^2\,cos^2\,k - b^2\,sin^2\,k}\right)^{\frac{1}{2}} dk.$$

The whole area inclosed by the curve is given by the value of $a\,b + (a^2 - b^2)\,tan^{-1}\frac{a}{b}$. The entire area of the lemniscate of Bernouilli is found by putting $a = b$ in the preceding expression, which gives the value a^2. The inverse of the curve from the center is the hyperbola

$$a^2\,x^2 - b^2\,y^2 = 1, \tag{20'}$$

which has the same eccentricity as the quartic. The polar reciprocal of this hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{21'}$$

showing that the inverse of an hyperbola from the center is the pedal of its polar reciprocal with respect to the center, as in the corresponding case of the ellipse.

The curve may be described mechanically by means of its pedal properties, very much as in the preceding case of the ellipse.

Many of the inverse theorems relating to the ellipse and oval are equally true when applied to the hyperbola and this curve, which may be called a lemniscate. Following are a few additional theorems relating more especially to the hyperbola and the lemniscate:

1. The difference of the two lines drawn from any point of an hyperbola to the foci is equal to the transverse axis.

2. The difference of the two lines drawn from any point on a lemniscate to the foci bears a constant ratio to the radius vector of the point.

1. *PN* is the ordinate of a point *P* on an hyperbola, *PG* is the normal meeting the axis in *G;* if *NP* be produced to meet the asymptote in *Q*, then is *QG* at right angles to the asymptote.

2. *PN* is a "central" circle through the point *P* on a lemniscate and whose "central diameter" is the axis of abscissæ, *PG* is a "central" circle cutting the curve orthogonally at *P* and meeting the axis of abscissæ at *G;* if the "central" circle *NP* cut the tangent at the origin at *Q*, then is the "central" circle *QR* cut orthogonally by this tangent.

The lemniscate may be said to have a "conjugate" lemniscate, just as the hyperbola has a conjugate hyperbola.

1. The equation of an hyperbola is $a^2x^2 - b^2y^2 = 1$. The equation of its conjugate is $a^2x^2 - b^2y^2 = -1$. The equation of its asymptote is $a^2x^2 - b^2y^2 = 0$. The equation of an hyperbola referred to its asymptotes is $xy = \dfrac{b^2 + a^2}{4a^2b^2}$. That of its conjugate is $xy = -\dfrac{b^2 + a^2}{4a^2b^2}$.

2. The equation of a lemniscate is $a^2x^2 - b^2y^2 = (x^2 + y^2)^2$. The equation of its conjugate is $a^2x^2 - b^2y^2 = -(x^2 + y^2)^2$. The equation of tangents at the origin is $a^2x^2 - b^2y^2 = 0$. The equation of a lemniscate referred to tangents at the origin is $xy = \dfrac{a^2 + b^2}{4a^2b^2}\,(x^2 + y^2)^2$. That of its conjugate is $xy = -\dfrac{a^2 + b^2}{4a^2b^2}\,(x^2 + y^2)^2$.

1. The two lines joining the points in which any two tangents to an hyperbola meet the asymptotes are parallel to the chord of contact of the tangents and are equidistant from it.

2. The two "central" circles joining the points in which any two "central"

tangent circles to a lemniscate meet the tangents at the origin are in directum with the "central" circle joining the points of contact of the two "central" tangent circles.

1. The asymptotes of an hyperbola coincide with the diagonals of the rectangle contained by the transverse and conjugate axes.

2. The tangents to a lemniscate through the origin coincide with the diagonals of the rectangle formed by the intersections of four "central" circles whose diameters are the four semi-axes of the curve respectively.

And in general, the inverse, with respect to the center, of a system of conics given by the equation,

$$a^2 x^2 + y^2 \frac{a^2}{1 - e^2} = 1, \tag{22}$$

is the system of pedal curves, whose equation is,

$$a^2 x^2 + y^2 \frac{a^2}{1 - e^2} = (x^2 + y^2)^2 \tag{3}$$

These curves belong in the general class of curves designated by the name of *bicircular quartics.*

1. The condition that the line whose equation is $y = mx + c$ shall touch the conic given by equation (22) is $c^2 = \frac{m^2}{a^2} + \frac{1 - e^2}{a^2}$

2. The condition that a "central" circle whose equation is $y = mx + c\,(x^2 + y^2)$ shall touch the pedal curve given by equation (3), is $c^2 = \frac{m^2 + 1 - e^2}{a^2}$

1. The equation of a tangent line at any point of the conic is $a^2 xx' + b^2 yy' = 1$.

2. The equation of the "central" tangent circle at any point of the pedal is $a^2 xx' + b^2 yy' = (x^2 + y^2)$.

In this manner many more sets of corresponding equations in the two systems of curves might be given; but the above examples are sufficient to show their general relations.

Following is a list of theorems relating to confocal conics and confocal pedal curves:

1. The equation of a system of confocal conics is

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1.$$

2. The equation of a system of confocal pedal curves is

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = (x^2 + y^2)^2.$$

1. Two conics of a confocal system pass through a point. One of these conics is an ellipse and the other an hyperbola.

2. Two pedal curves of a confocal system pass through a point. One of these curves is an oval and the other is a lemniscate.

1. One conic of a confocal system and only one will touch a given straight line.

2. One pedal curve of a confocal system and only one will touch a given "central" circle.

1. Two confocal conics cut one another at right angles at all their common points.

2. Two confocal pedal curves cut one another at right angles at all their common points.

1. The difference of the squares of the perpendiculars drawn from the center on any two parallel tangents to two given confocal conics is constant.
2. The difference of the squares of the reciprocals of the diameters of any two "central" tangent circles in directum with one another, to two given confocal pedal curves is constant.

1. If a tangent to one of two confocal conics be perpendicular to a tangent to the other, the locus of their point of intersection is a circle.
2. If a "central" circle tangent to one of two confocal pedal curves cut orthogonally a "central" tangent circle to the other, the locus of their point of intersection is a circle.

1. From any point *T* the two tangents *TP*, *TP'* are drawn to one conic, and the two tangents *TQ*, *TQ'* to a confocal conic; then will the straight lines *QP*, *Q'P* make equal angles with the tangent at *P*.
2. From any point *T* the two "central" tangent circles *TP*, *TP'* are drawn to one pedal curve, and the two "central" tangent circles *TQ*, *TQ'* to a confocal pedal curve; then will the "central" circles *QP*, *Q'P* cut the tangent at *P* at equal angles.

1. *TP*, *TQ* are tangents one to each of two fixed confocal conics; then, if the tangents are at right angles to one another the line *PQ* will always touch a third confocal conic.
2. *TP*, *TQ* are two "central" tangent circles one to each of two confocal pedal curves; then, if these circles cut one another orthogonally, the "central" circle *PQ* will always be tangent to a third confocal pedal curve.

1. If an ellipse have double contact with each of two confocal conics, the tangents at the points of contact will form a rectangle.
2. If an oval (concentric) have double contact with each of two confocal ovals, the "central" tangent circles at the points of contact will cut one another orthogonally.

1. A triangle circumscribes an ellipse and two of its angular points lie on a confocal ellipse; then will the third vertex lie on another confocal ellipse.
2. Three "central" circles are tangent internally to an oval, and two of their points of intersection lie on a confocal oval; then will the third point of intersection lie on another confocal oval.

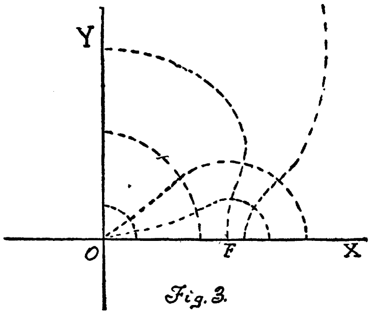

Fig. 3.

In figure 3 is shown the general appearance of a system of confocal pedal curves, consisting of ovals and lemniscates.

If the foregoing method of inversion be applied to a system of central conicoids, it gives rise to a system of surfaces of the fourth degree. These surfaces bear to the plane pedal curves just considered many of the relations that conicoids bear to plane conics They have three general forms according as the sections made by the coördinate planes are ovals or lemniscates, just as the central conicoids have the three forms of ellipsoid, hyperboloid of one sheet and hyperboloid of two sheets.

Let the general equation of a system of central conicoids be

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1, \tag{23}$$

where $a > b > c$, then that of the system of inverse surfaces will be

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = (x^2 + y^2 + z^2)^2. \tag{24}$$

This is the first pedal surface from the center of the conicoid given by the equation

$$a^2 x^2 \pm b^2 y^2 \pm c^2 z^2 = 1, \tag{25}$$

which in turn is the polar reciprocal with respect to the center, of the original conicoid. Hence, in general, the inverse of a conicoid from the center is the pedal surface of its polar reciprocal with respect to the center.

Resuming equation (24), it is evident that the three semi-axes of the surface are $\frac{1}{a}$, $\frac{\sqrt{\pm 1}}{b}$ and $\frac{\sqrt{\pm 1}}{c}$; and that the surface is symmetrical about any coordinate plane. The eccentricities of the sections made by the xy, xz, yz planes respectively are

$$e_z^2 = \frac{a^2 \mp b^2}{a^2},\ e_y^2 = \frac{a^2 \mp c^2}{a^2},\ e_x^2 = \frac{\pm b^2 \mp c^2}{\pm b^2} \tag{26}$$

When the signs in equation (24) are taken all positive, the surface is the pedal of an ellipsoid, and so far as this paper is concerned, may conveniently be called a pedal surface of the "first kind." It varies from a smooth to an indented surface, the transition occurring when either e_z or e_y or each becomes equal to $\sqrt{\tfrac{1}{2}}$. The two limiting forms are, a sphere when $e_z = e_y = e_x = 0$, and two spheres tangent at the origin to the xy plane when $e_y = e_x = 1$.

When the sign of $\frac{z^2}{c^2}$ in equation (24) is taken negative, the surface is the first pedal from the center of an hyperboloid of one sheet; and may be called a pedal surface of the "second kind."

The general form of the surface is represented by fig. 4.

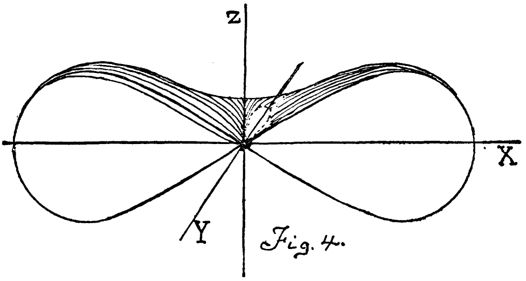

Fig. 4.

When the signs of $\frac{z^2}{c^2}$ and $\frac{y^2}{b^2}$, in equation (24) are taken both negative, the surface is the pedal from the center of an hyperboloid of two sheets; and may be called a pedal surface of the "third kind." The general form of the surface is shown in fig. 5; it is much the shape of two tops placed point to point.

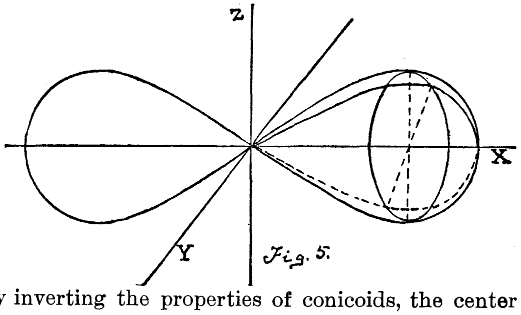

Fig. 5.

Following is a list of some of the more important theorems on the above pedal surfaces. They were obtained by inverting the properties of conicoids, the center of the conicoid being taken as the center of inversion. As in the case of plane pedal curves, only those properties were used which may be denominated "central properties"; that is, are dependent upon the center of the conicoid. The term "central sphere," analogous to "central circle," will be used to denote the inverse of a plane; it is a sphere passing through the origin or center of inversion.

1. All plane sections of a conicoid are conics.
2. All central plane sections of a pedal surface are plane pedal curves.

1. The equation of the tangent plane at any point of the conicoid,

$$ax^2 + by^2 + cz^2 + d = 0, \text{ is}$$
$$axx' + byy' + czz' + d = 0.$$

2. The equation of a central tangent sphere at any point of the pedal surface,

$$ax^2 + by^2 + cz^2 + d\ (x^2 + y^2 + z^2)^2 = 0, \text{ is}$$
$$axx' + byy' + czz' + d\ (x^2 + y^2 + z^2) = 0.$$

1. The condition that the plane $lx + my + nz + p = 0$ shall touch the above conicoid is $\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} + \frac{p^2}{d} = 0.$

2. The condition that the central tangent sphere $lx + my + nz + p\ (x^2 + y^2 + z^2) = 0$ shall touch the above pedal surface is,

$$\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} + \frac{p^2}{d} = 0.$$

1. The asymptotic cone to the conicoid given by equation (23) is

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 0.$$

2. The tangent cone at the origin to the pedal surface given by equation (24) is

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 0.$$

1. The condition that the conicoid shall be one of revolution is $(b - a)\ (c - a) = 0.$

2. The condition that the pedal surface shall be one of revolution is $(b - a)\ (c - a) = 0.$

1. The sum of the squares of the reciprocals of any three diameters of an ellipsoid which are mutually at right angles is constant.

2. The sum of the squares of any three diameters of a pedal surface of the "first kind" which are mutually at right angles is constant.

1. The locus of the point of intersection of three tangent planes to the conicoid given by equation (23) which are mutually at right angles is $x^2 + y^2 + z^2 = a^2 \pm b^2 \pm c^2$. This is the director sphere of the conicoid, and is real in the case of an ellipsoid; in the other cases it depends upon the values of a, b, and c.

2. The locus of the point of intersection of three central tangent spheres to the pedal surface given by equation (24) which cut one another orthogonally is $x^2 + y^2 + z^2 = \frac{1}{a^2 \pm b^2 \pm c^2}$. This may be called the director sphere of the pedal surface, and is always real in the case of a pedal surface of the "first kind."

1. The locus of the foot of a perpendicular from the center upon any tangent plane to a conicoid is a pedal surface.

2. The locus of the extremity of the "central" diameter of any "central" sphere tangent to a pedal surface is a conicoid. [By "central diameter" is meant the diameter which passes through the origin.]

1. Any tangent plane to the asymptotic cone of a conicoid meets the conicoid in two parallel straight lines, equidistant from the center.

2. Any tangent plane to the cone tangent at the origin to a pedal surface meets the surface in two equal circles in directum.

1. The area of any central plane section of an ellipsoid given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, is $\frac{\pi a b c}{(a^2 l^2+b^2 m^2+c^2 n^2)^{\frac{1}{2}}}$, the cutting plane being given by the equation $lx+my+nz=0$.

2. The area of any central plane section of a pedal surface of the "first kind" given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=(x^2+y^2+z^2)^2$, is $\frac{\pi}{2}(a b c)^{-2}[(b^2+c^2)a^2 l^2+(a^2+c^2)b^2 m^2+(a^2+b^2)c^2 n^2]$, the cutting plane being given by the equation $lx+my+nz=0$.

1. If central plane sections of an ellipsoid be of constant area, their planes touch a cone of the second degree.

2. If central plane sections of a pedal surface of the "first kind" be of constant area, their planes touch a cone of the second degree.

The six central planes cutting circular sections from the conicoid given by equation (23) are given by the three pairs of equations

$$y^2\left[\frac{1}{b^2}-\frac{1}{a^2}\right]+z^2\left[\frac{1}{c^2}-\frac{1}{a^2}\right]=0. \quad (i)$$

$$z^2\left[\frac{1}{c^2}-\frac{1}{b^2}\right]+x^2\left[\frac{1}{a^2}-\frac{1}{b^2}\right]=0. \quad (ii)$$

$$x^2\left[\frac{1}{a^2}-\frac{1}{c^2}\right]+y^2\left[\frac{1}{b^2}-\frac{1}{c^2}\right]=0. \quad (iii) \qquad (27)$$

Taking the case of the ellipsoid, the two systems of real circular sections are given by the equations

$$x\left[\frac{1}{b^2}-\frac{1}{a^2}\right]^{\frac{1}{2}}\pm z\left[\frac{1}{c^2}-\frac{1}{b^2}\right]^{\frac{1}{2}}=p\left[\frac{1}{c^2}-\frac{1}{b^2}\right]^{\frac{1}{2}} \qquad (28)$$

where p is the perpendicular distance from the origin upon the cutting plane. Since a circle in space may always be considered as the intersection of a sphere by a plane, it follows that the inverse of a circle from any point is also a circle. Hence, the inverse of a system of circular sections of a conicoid is a system of circular sections of a pedal surface. And it is seen from equations (27) and (28) above, that there are two such systems of real circular sections in each of the three pedal surfaces. In the case of a pedal surface of the "first kind," these sections all lie on the system of central spheres which are bisected by the xz plane in the circles,

$$x\left[\frac{1}{b^2}-\frac{1}{a^2}\right]^{\frac{1}{2}}\pm z\left[\frac{1}{c^2}-\frac{1}{b^2}\right]^{\frac{1}{2}}=p\left[\frac{1}{c^2}-\frac{1}{a^2}\right]^{\frac{1}{2}}(x^2+z^2). \qquad (29)$$

When the sphere given by equation (29) is tangent to the surface, the point of tangency may properly be called an umbilicus, corresponding to an umbilicus in the conicoid. Hence the pedal surface has four umbilici, that is, points at which tangent planes will cut out infinitely small circles.

Following are a few additional theorems on circular sections:

1. In a conicoid, any two circular sections of opposite systems are on a sphere.

2. In the pedal surface, any two circular sections of opposite systems are on a sphere.

1. If the squares of the axes of an ellipsoid are in arithmetical progression the umbilici lie on the central circular sections: if they are in harmonic progres-

sion, the circular sections are at right angles: if they are in geometrical progression the tangent planes at the umbilici touch the sphere through the central circular sections.

2. If the squares of the axes of a pedal surface of the "first kind" are in harmonical progression, the umbilici lie on the central circular sections: if they are in arithmetical progression, the two systems of spheres cutting the circular sections have their "central diameters" at right angles: if they are in geometrical progression, the central tangent spheres at the umbilici touch the sphere through the central circular sections.

1. The angle made by the two systems of planes cutting circular sections from an ellipsoid is $k = 2 \tan^{-1} \frac{c}{a} \left\{\frac{b^2-a^2}{c^2-b^2}\right\}^{\frac{1}{2}}$.

2. The angle between the "central diameters" of the two systems of central spheres cutting circular sections from a pedal surface of the "first kind" is $k = 2 \cot^{-1} \frac{c}{a} \left\{\frac{b^2-a^2}{c^2-b^2}\right\}^{\frac{1}{2}}$.

Since a straight line in space inverts into a circle through the origin, the straight-line generators of a conicoid invert into circle generators of a pedal surface. Hence the properties of ruled conicoids, when inverted, give rise to corresponding properties of circularly ruled pedal surfaces. This circular generator lies entirely in the surface and passes through the origin. And since the hyperboloid of one sheet is the only central conicoid having straight-line generators, the pedal surface of the "second kind" is the only one of the three surfaces under discussion having circular generators. Also, corresponding to the two systems of straight-line generators of the hyperboloid are two systems of circular generators of the pedal surface. Following are a few theorems on the pedal surface of the "second kind" obtained by inverting those properties of the hyperboloid of one sheet which relate to straight-line generators:

1. The tangent plane to an hyperboloid of one sheet at a point through which a generating line passes will contain that generating line.

2. The central tangent sphere to a pedal surface of the "second kind" at a point through which a generating circle passes will contain that circle.

1. Through any point on a generating line of an hyperboloid of one sheet another generating line passes, and they are both in the tangent plane at the point.

2. Through any point on a generating circle of a pedal surface of the "second kind" another generating circle passes, and they are both in the central tangent sphere at the point.

1. Any plane through a generating line of a conicoid touches the surface, its point of contact being the point of intersection of the two generating lines which lie upon it.

2. Any central sphere through a generating circle of a pedal surface touches the surface, its point of contact being the point of intersection of the two generating circles which lie upon it.

1. Three non-intersecting generators are sufficient to determine the conicoid on which they lie.

2. Three non-intersecting (except, of course, at the origin) circular generators are sufficient to determine the pedal surface on which they lie.

1. The straight lines which intersect three fixed non-intersecting straight lines are generators of the same system of a conicoid, and the three fixed lines are generators of the opposite system of the same conicoid.

2. The central circles which intersect three fixed non-intersecting central circles are generating circles of the same system of a pedal surface, and the three fixed circles are generating circles of the opposite system of the same pedal surface.

1. Two straight lines, and two only, will, in general, meet each of four given non-intersecting straight lines; but if the four given straight lines are all generators of the same system of a conicoid, then an infinite number of straight lines will meet the four, which will all be generators of the opposite system of the same conicoid.

2. Two central circles, and two only, will, in general, meet each of four given non-intersecting central circles; but if the four given circles are all generating circles of the same system of a pedal surface, then an infinite number of central circles will meet the four, which will all be generating circles of the opposite system of the same pedal surface.

1. Two planes are drawn, one through each of two intersecting generating lines of a conicoid; these planes meet the surface in two other intersecting generating lines.

2. Two central spheres are drawn, one through each of two intersecting generating circles of a pedal surface; these spheres meet the surface in two other intersecting generating circles.

1. The plane through the center of a conicoid and any generating line will cut the surface in a parallel generating line, and will touch the asymptotic cone.

2. The plane through the center of a pedal surface and any generating circle will cut the surface in another generating circle equal to the first, and having a common diameter with it, and will touch the tangent cone at the origin.

1. The locus of the point of intersection of perpendicular generators of an hyperboloid of one sheet is a sphere $x^2 + y^2 + z^2 = a^2 + b^2 - c^2$.

2. The locus of the point of intersection of generating circles of a pedal surface of the "second kind" that cut orthogonally is a sphere given by the equation $x^2 + y^2 + z^2 = (a^2 + b^2 - c^2)^{-1}$.

This and the following theorem depend upon the fact that an angle in space inverts into an equal angle.

1. The angle between the generating lines through the point (x, y, z) of the quadric, $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1$, is $\cos^{-1} \frac{h_1 + h_2}{h_1 - h_2}$, where h_1, h_2, are the roots of the equation, $\frac{x^2}{a(a-h)} + \frac{y^2}{b(b+h)} + \frac{z^2}{c(c+h)} = 0$.

2. The angle between the generating circles through the point (x, y, z) of the quartic, $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = (x^2 + y^2 + z^2)^2$, is $\cos^{-1} \frac{h_1 - h_2}{h_1 - h_2}$, where h_1, h_2, are roots of the equation $\frac{x^2}{a(a-h)} + \frac{y^2}{b(b+h)} + \frac{z^2}{c(c+h)} = 0$.

If a system of confocal conicoids be inverted from the center, a system of confocal pedal surfaces is obtained. The general equation of such a system is

$$\frac{x^2}{a^2 + h} + \frac{y^2}{b^2 + h} + \frac{z^2}{c^2 + h} = (x^2 + y^2 + z^2)^2 \qquad (30)$$

If h is positive, the surface is a pedal of the "first kind," whose axes decrease as k increases: so that the limiting form of the surface when $k = \infty$ is a point sphere at the origin.

If k is negative and less than c^2, the surface is a pedal of the "first kind," which recedes more and more from the spherical form, like an hour-glass, until, when $k = -c^2$, it folds over on itself, as it were, and embraces all of the xy plane exterior to the oval whose equations are

$$\frac{x^2}{a^2-c^2} + \frac{y^2}{b^2-c^2} = (x^2+y^2)^2,\ z = 0. \tag{31}$$

When k is between $-c^2$ and $-b^2$, the surface is a pedal of the "second kind" which approaches as a limit, when k is very nearly equal to $-c^2$ that portion of the xy plane inclosed by the oval given by equation (31); and when k is very nearly equal to $-b^2$, the surface is very nearly coincident with that part of the xz plane which is exterior to the lemniscate whose equations are

$$\frac{x^2}{a^2-b^2} + \frac{z^2}{c^2-b^2} = (x^2+y^2)^2,\ y = 0. \tag{32}$$

If k is between $-b^2$ and $-a^2$, the surface is a pedal of the "third kind." When k is very nearly equal to $-b^2$, the surface is very nearly coincident with that portion of the $x\,z$ plane which is inclosed by the lemniscate given by equation (32). When $y = -a^2$, the surface becomes imaginary; but analytically it is all that portion of the $y\,z$ plane exterior to the imaginary curve

$$\frac{y^2}{b^2-a^2} + \frac{z^2}{c^2-a^2} = (y^2+z^2)^2,\ x = 0. \tag{33}$$

And when k is between $-a^2$ and $-\infty$ the surface continues imaginary. The two pedal curves given by equations (31) and (32) may be called the focal pedal curves of the system, corresponding in their properties to the focal conics of a system of confocal conicoids. Following is a list of corresponding theorems on confocal conicoids and confocal pedal surfaces:

1. Three conicoids of a confocal system pass through any given point: one of them is an ellipsoid, one an hyperboloid of one sheet, and one an hyperboloid of two sheets.

2. Three pedal surfaces of a confocal system pass through any given point: one of them is a pedal surface of the "first kind," one is of the "second kind," and one is of the "third kind."

1. One conicoid of a given confocal system will touch any plane.
2. One pedal surface of a confocal system will touch any central sphere.

1. Two conicoids of a confocal system will touch any straight line.
2. Two pedal surfaces of a confocal system will touch any central circle.

1. Two confocal conicoids cut one another at right angles at all their common points.

2. Two confocal pedal surfaces cut one another orthogonally at all their common points.

1. If a straight line touch two confocal conicoids, the tangent planes at the points of contact will be at right angles.

2. If a central circle touch two confocal pedal surfaces, the central tangent spheres at the points of contact will cut one another orthogonally.

1. The difference of the squares of the perpendiculars from the center on any two parallel tangent planes to two given confocal conicoids is constant.

2. The difference of the squares of the reciprocals of the diameters of any two

central tangent spheres in directum to two given confocal pedal surfaces is constant.

1. The locus of the point of intersection of three planes mutually at right angles, each of which touches one of three given confocal conicoids, is a sphere.
2. The locus of the point of intersection of three central spheres cutting one another orthogonally, each of which touches one of three given confocal pedal surfaces, is a sphere.

1. The locus of the umbilici of a system of confocal ellipsoids is the focal hyperbola.
2. The locus of the umbilici of a system of confocal surfaces of the "first kind" is the focal lemniscate.

1. If two concentric and co-axial conicoids cut one another everywhere at right angles they must be confocal.
2. If two concentric and co-axial pedal surfaces cut one another everywhere at right angles they must be confocal.

1. Three confocal conicoids meet in a point, and a central plane of each is drawn parallel to its tangent plane at that point, then, one of the three sections will be an ellipse, one an hyperbola, and one imaginary.
2. Three confocal pedal surfaces meet in a point, and a central plane of each is drawn tangent at the origin to the central sphere tangent at that point, then, one of the three sections will be an oval, one a lemniscate, and one imaginary.

1. If three lines at right angles to one another touch a conicoid, the plane through the points of contact envelop a confocal conicoid.
2. If three central circles at right angles to one another touch a pedal surface, the central sphere through the points of contact will always touch a confocal pedal surface.

Many other theorems relating to the curves and surfaces under discussion might be obtained; but those already given show their principal properties, and more would merely add comparatively uninteresting details.